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New nonextensive quantum entropy and strong evidences for equilibrium of quantum hadronic states

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Abstract

In this Letter by introduction of the nonextensivity conjugated entropy $\overline{S}_{J\theta}(p,q)$ we proved new nonextensive entropic uncertainty relations (NC-EUR) as well as new entropic uncertainty bands (NC-EUB). Moreover, we obtained not only consistent experimental tests of the optimal NC-EUR and NC-EUB, but also strong experimental evidences for the principle of minimum distance in space of quantum states (PMD-SQS).

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1. Introduction

It is well known that in many places of science there are phenomena which clearly indicate the existence of some degree of nonextensivity in the thermostatistical sense. They include all situations characterized by long-range interactions, long-range microscopic memory effects as well as multi-fractal space–time (or multi-fractal phase space) structures of the phenomena. In general, in hadronic systems the nonextensivity is expected to emerge due to the presence of the quarks and gluons having a strong-coupling long-range regime of quantum chromodynamics. The pion–nucleon and pion–nucleus quantum scattering systems are the most suitable for such investigations since in these cases the experimental nonextensive entropies can be obtained with high accuracy from the available phase shifts analyses. So, in the last time the investigation the quantum entropy [1] was substantially extended not only by proving new entropic uncertainty relations (EUR) (see, e.g., Refs. [2]) for the standard additive systems but also by generalization of such results to the nonextensive statistics [3]. The possibility of utilizing the Tsallis entropy [3] as a measure of quantum uncertainty is discussed in Refs. [4–13]. Entropic upper bounds for each Tsallis-like scattering [$S_\theta(q)$, $S_L(q)$, $S_{\theta L}(q)$]-entropies were recently [7] proved in terms of optimal scattering entropies derived from the principle of minimum distance in the space of states (PMD-SQS) [12]. In this way the entropic uncertainty bands (EUB) were introduced in Ref. [7] while the principle of limited entropic uncertainty has been developed in Ref. [9]. Moreover, by experimental determination of the (p,q)-nonextensivity indices from the pion–nucleus and pion–nucleon phase shifts we obtained [10,11] strong experimental evidences for the $(1/2p + 1/2q = 1)$-nonextensivity conjugation (NC) for (J,θ)-nonextensive quantum statistics with $0.50 \leqslant p \leqslant 0.60$ and $q = p/(2p-1)$. How-

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ever, the best fit (CL > 99%) is obtained only for the $[S_J(p), S_\theta(q)]$-scattering entropies (see Fig. 1 in the second Ref. [10]) while the high departure from the (PMD-SQS)-equilibrium predictions are observed for joint $S_{J\theta}(p)$-scattering entropies. These results strongly suggested us that a new joint entropy is necessary to be defined as a measure of quantum uncertainty when the nonextensivity conjugation is taken into account.

In this Letter, by introduction of a new scattering entropy $\overline{S}_{J\theta}(p,q)$, called nonextensivity conjugated (NC) entropy, we prove new entropic uncertainty relations (EUR) [see the NC-EUR (lower bound) (25)], and also new entropic uncertainty bands [see the NC-EUB (25)]. Moreover, results of experimental tests of the optimal NC-EUR and NC-EUB, obtained by using 88-sets of experimental pion–nucleon phase shifts, will be presented.

2. Definition of nonextensivity conjugated joint entropy

In order to make our presentation self-contained we shall first provide the basic formulas for the quantum nonextensive scattering entropies which are necessary for our investigation. Of course, the results can be presented in a general way for particles with arbitrary spins, but, for sake of simplicity as well as for direct experimental tests, we consider here only the hadronic scattering process

$$0^- + \frac{1}{2}^+ \to 0^- + \frac{1}{2}^+. \tag{1}$$

The helicity amplitudes $f_{++}(x)$ and $f_{+-}(x)$ are normalized such that the c.m. differential cross section $d\sigma/d\Omega(x)$ $[x = \cos\theta, \in (+1,-1), \theta$-being the c.m. scattering angle] is defined by

$$\frac{d\sigma}{d\Omega}(x) = |f_{++}(x)|^2 + |f_{+-}(x)|^2 \tag{2}$$

and that the elastic integrated cross section σ_{el} is given by

$$\sigma_{\text{el}} = 2\pi \int_{-1}^{+1} \frac{d\sigma}{d\Omega}(x)\,dx$$

$$= 2\pi \int_{-1}^{+1} \left[|f_{++}(x)|^2 + |f_{+-}(x)|^2\right] dx. \tag{3}$$

So, if the angular probability distribution $P(x)$ is given by

$$P(x) = \frac{2\pi}{\sigma_{\text{el}}} \cdot \frac{d\sigma}{d\Omega}(x), \qquad \int_{-1}^{1} P(x)\,dx = 1 \tag{4}$$

then, the Tsallis-like angular entropies $S_\theta(q)$ which allow us to investigate the (extensive or nonextensive)-statistical behavior of the $[\theta]$-quantum scattering states considered as canonical ensembles, can be defined as follows

$$S_\theta(q) = \left[1 - \int_{-1}^{1} dx\,\left[P(x)\right]^q\right]\frac{1}{q-1}, \quad q \in R,$$

$$\lim_{q\to 1} S_\theta(q) = S_\theta(1) = -\int_{-1}^{1} dx\, P(x) \ln P(x). \tag{5}$$

Now, if the helicities of incoming and outgoing nucleons are denoted by μ, μ', and was written as $(+)$, $(-)$, corresponding to $(\frac{1}{2})$ and $(-\frac{1}{2})$, respectively. The helicity amplitudes, in terms of the partial waves amplitudes f_{J+} and f_{J-}(see Ref. [13]), we have

$$f_{++}(x) = \sum_{J=\frac{1}{2}}^{J_{\max}} (J+1/2)(f_{J-} + f_{J+}) d^J_{\frac{1}{2}\frac{1}{2}}(x), \tag{6}$$

$$f_{+-}(x) = \sum_{J=\frac{1}{2}}^{J_{\max}} (J+1/2)(f_{J-} - f_{J+}) d^J_{-\frac{1}{2}\frac{1}{2}}(x), \tag{7}$$

where $d^J_{\mu\nu}(x)$ are the usual rotation functions [see below Eqs. (16), (17)].

Now, the elastic integrated cross section (3) for the meson–nucleon scattering can be expressed in terms of partial amplitudes f_{J+} and f_{J-}

$$\frac{\sigma_{\text{el}}}{2\pi} = \int_{-1}^{+1} dx\, \frac{d\sigma}{d\Omega}(x)$$

$$= \sum_{J=\frac{1}{2}}^{J_{\max}} (2J+1)\left(|f_{J+}|^2 + |f_{J-}|^2\right). \tag{8}$$

J-nonextensive statistics for the quantum scattering: we define two kind of Tsallis-like scattering entropies. One of them, namely $S_J(p)$, $p \in R$, is special dedicated to the investigation of the nonextensive statistical behavior of the angular momentum J quantum states, and can be defined by

$$S_J(p) = \left[1 - \sum_{l=0}^{J}(2J+1)\cdot[p_j]^p\right]\frac{1}{p-1}, \quad p \in R,$$

$$\lim_{p\to 1} S_J(p) = S_J(1) = -\sum_{l=0}^{J}(2J+1)\cdot p_J \cdot \ln p_J, \quad (9)$$

where the probability distributions p_J, is given by

$$p_J = \frac{|f_{J+}|^2 + |f_{J-}|^2}{\sum_{J=\frac{1}{2}}^{J_{\max}}(2J+1)(|f_{J+}|^2 + |f_{J-}|^2)},$$

$$\sum_{J=\frac{1}{2}}^{J_{\max}}(2J+1)p_J = 1. \quad (10)$$

It it important to note that the scattering entropies (5) and (9) can be interpreted alternatively as natural measures of the uncertainties corresponding to the scattering distributions $P(x)$ and p_l, respectively. Moreover, if we are interested to obtain a measure of uncertainty of the simultaneous realization of the probability distributions $P(x)$ and p_l, then, we must introduce the joint entropies corresponding to the products of these distributions: $p_l \cdot P(x)$. Consequently, in this case the Tsallis-like scattering entropies for the scattering of spinless particles are defined by:

$$S_{J\theta}(q) = \left\{1 - \sum_{l=0}^{J}(2l+1)p_l^q \int_{-1}^{1} dx\,\left[P(x)\right]^q\right\}\frac{1}{q-1}.$$

Next, in Refs. [10,11] we investigated the nonextensivities of quantum statistics of the hadronic scattering states by using the $[S_J(p), S_\theta(q)]$-scattering entropies, corresponding to the systems of the angular momentum states [J-states] and the quantum angular scattering states [θ-states], respectively. Then, by experimental determination of the (p,q)-nonextensivity indices from the pion–nucleus and pion–nucleon phase shifts we obtained that the good fits are obtained with $p \in (1/2, 1)$ and $q \in (1, \infty)$. Such results strongly suggested us to look at Fourier transformation (6), (7) as to a bounded map T from the space L_{2p} of the "partial" J-amplitudes to the space L_{2q} of the helicity amplitudes. In this case we proved the following nonextensivity conjugation principle:

Nonextensivity conjugation principle (see the proofs in Refs. [10,11]): If the Fourier transform (6), (7) is a bounded map $T : L_{2p} \to L_{2q}$, then the $[p, J]$- and $[q,\theta]$-nonextensive quantum statistics are correlated via the relation: $\frac{1}{2p} + \frac{1}{2q} = 1$, or $q = p/(2p-1)$, while the norm estimate M of this map is given by: $||Tf||_{2q}/||f||_{2p} \leqslant 2^{\frac{p-1}{2p}} = 2^{\frac{1-q}{2q}}$.

This principle is confirmed experimentally with high accuracy in both pion–nucleon [10] and pion–nucleus [11] scattering cases.

So, in order to include the nonextensivity conjugation principle [10,11], is naturally to define a new scattering joint entropy $\overline{S}_{J\theta}(p,q)$ by:

$$\begin{aligned}
&\overline{S}_{J\theta}(p,q) \\
&\quad = \left[1 - \sum_{l=0}^{J}(2l+1)p_l^p \int_{-1}^{1} dx\,\left[P(x)\right]^q\right]\frac{1}{p-1} \\
&\quad = S_J(p) + \overline{S}_\theta(p) + (1-p)\overline{S}_\theta(p)S_J(p) = \overline{S}_{J\theta}(p); \\
&\overline{S}_\theta(p) \equiv \frac{1-q}{1-p}S_\theta(q), \\
&\quad p \in (1/2, \infty) \text{ and } q = p/(2p-1), \\
&\lim_{p\to 1} \overline{S}_{J\theta}\big[p, p/(2p-1)\big] = S_J(1) + \overline{S}_\theta(1) \\
&\qquad = S_J(1) - S_\theta(1). \qquad (11)
\end{aligned}$$

Therefore, in order to include the nonextensivity conjugation we must work with the $[S_J(p), \overline{S}_\theta(p), \overline{S}_{J\theta}(p)]$ system of scattering entropies instead of the $[S_J(p), S_\theta(p), S_{J\theta}(p)]$ scattering entropies.

3. PMD-SQS-optimal quantum-equilibrium limit

We next consider the maximum-entropy (MaxEnt) problems

$$\max\big\{\overline{S}_{J\theta}(p), S_J(p), \overline{S}_\theta(q)\big\} \quad \text{when } \sigma_{\text{el}} = \text{fixed}$$
$$\text{and } \frac{d\sigma}{d\Omega}(1) = \text{fixed} \qquad (12)$$

as criterion for the determination of the equilibrium distributions p_J^{me} and $P^{\text{me}}(x)$ for the ensemble of quantum states from the $(0^-\frac{1}{2} \to 0^-\frac{1}{2})$ scattering.

The equilibrium (optimal) distributions, as well as the optimal scattering entropies for the quantum scattering of the spinless particles was obtained in Refs. [9,11] (see Eqs. (21)–(25)). For the J-quantum states, in the spin $(0^-\frac{1}{2} \to 0^-\frac{1}{2})$ scattering case, these distributions are given by:

$$p_J^{\mathrm{me}} = p_J^{o1} = \frac{1}{2K_{\frac{1}{2}\frac{1}{2}}(1,1)} = \frac{1}{(J_o+1)^2 - 1/4} \quad \text{for } \frac{1}{2} \leqslant J \leqslant J_o, \text{ and}$$
$$p_J^{\mathrm{me}} = 0 \quad \text{for } J \geqslant J_o + 1 \tag{13}$$

while, for the θ-quantum states, these distributions are as follows:

$$P^{\mathrm{me}}(x) = P^{o1}(x) = \left[\frac{K^2_{\frac{1}{2}\frac{1}{2}}(x,1)}{K_{\frac{1}{2}\frac{1}{2}}(1,1)}\right], \tag{14}$$

where the reproducing kernel $K_{\frac{1}{2}\frac{1}{2}}(x,y)$ is given by

$$K_{\frac{1}{2}\frac{1}{2}}(x,y) = \frac{1}{2}\sum_{1/2}^{J_o}(2J+1)d^J_{\frac{1}{2}\frac{1}{2}}(x)d^J_{\frac{1}{2}\frac{1}{2}}(y),$$
$$K_{\frac{1}{2}\frac{1}{2}}(1,1) = \frac{1}{2}\left[(J_o+1)^2 - 1/4\right] = \frac{2\pi}{\sigma_{\mathrm{el}}}\frac{d\sigma}{d\Omega}(1), \tag{15}$$

where $d^J_{\frac{1}{2}\frac{1}{2}}(x)$ are the $d_{\mu\nu}$-spin rotation functions for the spin 1/2 particles,

$$d^J_{\frac{1}{2}\frac{1}{2}}(x) = \frac{1}{l+1}\cdot\left[\frac{1+x}{2}\right]^{\frac{1}{2}}\left[\mathring{P}_{l+1}(x) - \mathring{P}_l(x)\right], \tag{16}$$
$$d^J_{-\frac{1}{2}\frac{1}{2}}(x) = \frac{1}{l+1}\cdot\left[\frac{1-x}{2}\right]^{\frac{1}{2}}\left[\mathring{P}_{l+1}(x) + \mathring{P}_l(x)\right] \tag{17}$$

and $\mathring{P}_l(x) \equiv dP_l(x)/dx$.

Proof. In this case solving the problem (12) via Lagrange multipliers we obtain that the singular solution $\lambda_0 = 0$ exists and is just given by the $[\overline{S}^{o1}_{J\theta}]$ optimal entropies corresponding to the PMD-SQS optimal state:

$$f^{o1}_{++}(x) = f_{++}(1)K_{\frac{1}{2}\frac{1}{2}}(x,1)/K_{\frac{1}{2}\frac{1}{2}}(1,1),$$
$$f^{o1}_{+-}(x) = 0. \tag{18}$$

Indeed, the problem (16) is equivalent to the following unconstrained maximization problem:

$$\pounds \to \max, \tag{19}$$

where the Lagrangian function is defined as

$$\begin{aligned}\pounds \equiv{}& \lambda_0\left\{\overline{S}_{J\theta}(p), S_J(p), \overline{S}_\theta(p)\right\} \\ &+ \lambda_1\left\{\frac{\sigma_{\mathrm{el}}}{4\pi} - \sum(2J+1)\left[|f_{J-}|^2 + |f_{J+}|^2\right]\right\} \\ &+ \lambda_2\left\{\frac{d\sigma}{d\Omega}(1) - \left[\sum(J+1/2)\,\mathrm{Re}(f_{J+}+f_{J-})\right]^2\right. \\ &\left. - \left[\sum(J+1/2)\,\mathrm{Im}(f_{J+}+f_{J-})\right]^2\right\}.\end{aligned} \tag{20}$$

Hence, the solution of the problem (12) in the singular case $\lambda_0 = 0$ is reduced just to the solution of the minimum constrained distance in space of quantum states:

$$\sum(2J+1)\left[|f_{J-}|^2 + |f_{J+}|^2\right] \quad \text{when } \frac{d\sigma}{d\Omega}(1) \text{ is fixed.} \tag{21}$$

Therefore, by a straightforward calculus we obtain that the solution of the problem (12) is given by

$$S_J^{\max}(p) = S_J^{o1}(p) = \left[1 - \left[2K_{\frac{1}{2}\frac{1}{2}}(1,1)\right]^{1-p}\right]\frac{1}{p-1}, \tag{22}$$

$$S_\theta^{\max}(q) = S_\theta^{o1}(q) = \left[1 - \int_{-1}^{+1}dx\left(\frac{\left[K_{\frac{1}{2}\frac{1}{2}}(x,1)\right]^2}{K_{\frac{1}{2}\frac{1}{2}}(1,1)}\right)^q\right]\frac{1}{q-1}, \tag{23}$$

$$\overline{S}^{\max}_{J\theta}(p) = \overline{S}^{o1}_{J\theta}(p) = \left[1 - \left[2K_{\frac{1}{2}\frac{1}{2}}(1,1)\right]^{1-p} \times \int_{-1}^{+1}dx\left(\frac{\left[K_{\frac{1}{2}\frac{1}{2}}(x,1)\right]^2}{K_{\frac{1}{2}\frac{1}{2}}(1,1)}\right)^q\right]\frac{1}{p-1}, \tag{24}$$

where the reproducing kernels $K_{\frac{1}{2}\frac{1}{2}}(x,1)$ are given by Eq. (15).

4. Nonextensivity conjugated entropic uncertainty bands (NC-EUB)

Now, we prove the following main results.

Nonextensivity conjugated entropic uncertainty bands (NC-EUB). If σ_{el} and $\frac{d\sigma}{d\Omega}(1)$ are known from experiment then the nonextensivity conjugated entropy $\overline{S}_{J\theta}(p,q)$, defined by Eq. (11), must obey the following *optimal band*

$$\overline{S}^{\min}_{J\theta}(p,q) \leqslant \overline{S}_{J\theta}(p,q) \leqslant \overline{S}^{o1}_{J\theta}(p,q),$$
$$\overline{S}^{\min}_{J\theta}(p,q) = \left[1 - 2^{1-p}\int_{-1}^{+1}\left(\frac{[K_{\frac12\frac12}(x,1)]^2}{K_{\frac12\frac12}(1,1)}\right)^q dx\right]\frac{1}{p-1},$$
$$\overline{S}^{o1}_{J\theta}(p) = \left[1 - [2K_{\frac12\frac12}(1,1)]^{1-p} \times \int_{-1}^{+1} dx \left(\frac{[K_{\frac12\frac12}(x,1)]^2}{K_{\frac12\frac12}(1,1)}\right)^q\right]\frac{1}{p-1} \tag{25}$$

for any $p \in (\frac12, \infty)$ and $q = p/(2p-1)$, where the reproducing kernel $K_{\frac12\frac12}(x,1)$ and $\overline{S}^{o1}_{J\theta}(p)$ are given by Eqs. (15) and (24), respectively.

Proof. The proof of the upper bound (25) was given in preceding section by Eq. (24) while the proof of the lower bound (25) can be based on the following entropic inequalities

$$S^{\min}_J(t) \leqslant S_J(t) \leqslant S^{o1}_J(t),$$
$$S_\theta(t) \leqslant S^{o1}_\theta(t) \quad \text{for } t > 0 \tag{26}$$

which also was proved in the preceding section, where $S^{\min}_J(t) = S^{\min}_J(p) = \frac{1}{p-1}[1 - 2^{1-p}]$ (see Ref. [10]).

Now, let p and q be defined as: $\frac12 < p \leqslant 1$ and $1 \leqslant q < \infty$. In this case the bounds (26) are equivalent to

$$1 + (1-p)S^{\min}_J(p) \leqslant 1 + (1-p)S_J(p) \leqslant 1 + (1-p)S^{o1}(p),$$
$$1 + (1-p)S_\theta(p) \leqslant 1 + (1-p)S^{o1}(p), \tag{27}$$

$$1 + (1-q)S^{o1}_J(q) \leqslant 1 + (1-q)S_J(q) \leqslant 1 + (1-p)S^{\min}_J(p),$$
$$1 + (1-q)S^{o1}_\theta(q) \leqslant 1 + (1-q)S_\theta(q). \tag{28}$$

Hence, by multiplication of the second inequality (27) with the first inequalities (28) we get

$$[1 + (1-p)S^{\min}_J(p)][1 + (1-q)S^{oy}_\theta(q)] \leqslant [1 + (1-p)S_J(p)][1 + (1-q)S_\theta(q)], \quad \text{or}$$

$$\{[1 + (1-p)S^{\min}_J(p)] \times [1 + (1-q)S^{oy}_\theta(q)] - 1\}/(1-p) \leqslant \overline{S}_{J\theta}(p,q). \tag{29}$$

Hence, with (29) we proved (25) for the case $\frac12 < p \leqslant 1$ and $1 \leqslant q < \infty$, since $S^{\min}_J(p) = \frac{1}{p-1}[1 - 2^{1-p}]$. In similar way, the result (25) for $\frac12 < q \leqslant 1$ and $1 \leqslant p < \infty$ can be obtained by multiplication of the first inequality (27) with the second inequalities (28).

5. Experimental tests of the optimal NC-entropic uncertainty bands

Next, for numerical investigation of the optimal NC-EUB (25) we calculated the $[\overline{S}_{J\theta}(p,q), S_{J\theta}(p)]$-nonextensive scattering entropies by using the available experimental phase-shifts [14] for the $[\pi^+ p, \pi^- p, \pi^0 p]$ scatterings. So, in Figs. 1–3 we present the results of the first experimental test of NC-EUB (25) in pion–nucleus scatterings for three kinds of nonextensivity conjugated statistics: (a) $[p = 2/3, q = 2]$, (b) $[p = 1, q = 1]$ and (c) $[p = 2.0, q = 2/3]$, respectively. So, the experimental results for the nonextensive conjugated entropies $\overline{S}_{J\theta}(p,q)$ are compared with the results of the PMD-SQS optimal state predictions [12] for the $[\overline{S}^{\min}_{J\theta}(q), \overline{S}^{o1}_{J\theta}(p,q)]$ NC-EUB (25). The grey regions from Figs. 1–4 are obtained by assuming a minimum error of $\Delta J_o = \pm 1$ in the estimation of the optimal angular momentum

$$J_o = \text{half integer}\left[\frac{4\pi}{\sigma_{\text{el}}}\frac{d\sigma}{d\Omega}(1) + \frac14\right]^{1/2} - 1$$

from the experimental data [14]. From Figs. 1–3 we see that the experimental data on the nonextensivity conjugated entropies $\overline{S}_{J\theta}(p,q)$ are in excellent agreement (CL > 99%) with the [$\overline{S}^{o1}_{J\theta}(q)$] optimal (equilibrium) state predictions if the nonextensivities p and q of the (J and θ) statistics are considered correlated via the Riesz–Thorin relation: $\frac{1}{2p} + \frac{1}{2q} = 1$ (or $q = p/(2p-1)$). So, the best fit is obtained for the conjugate pairs p and $q = p/(2p-1)$ with the values of p in the range $1/2 \leqslant p \leqslant 2/3$. In fact a significant departures from "optimality" is observed only for pairs ($p \geqslant 2$ and $1/2 \leqslant q \leqslant 2/3$) where $\chi^2/n_D \geqslant 500$. These results for $\overline{S}_{J\theta}(p,q)$ entropy can be compared with those for $S_{J\theta}(p)$ nonextensive scattering

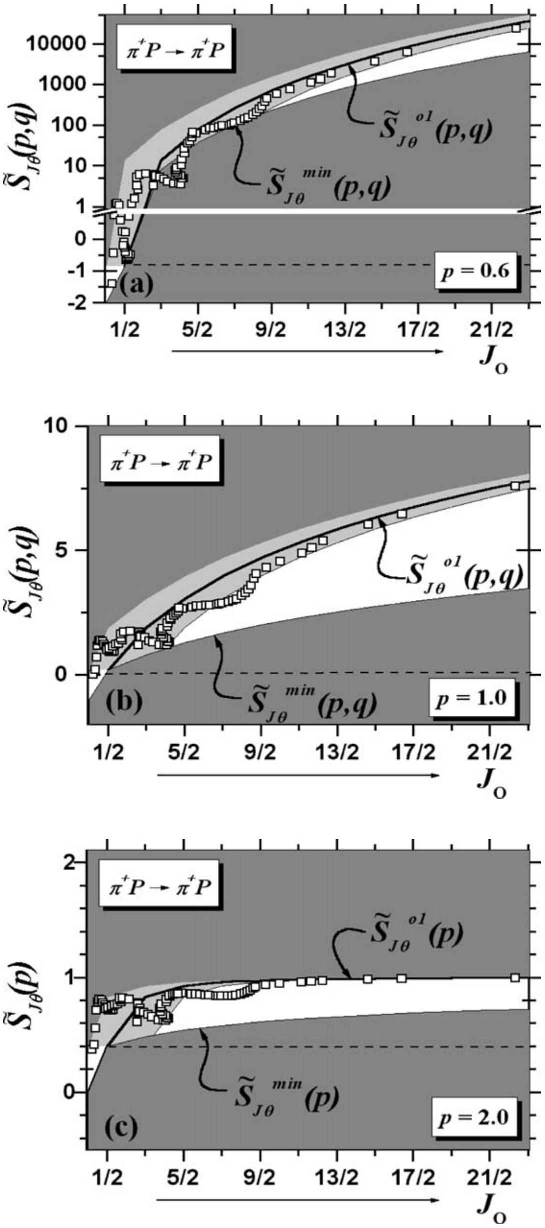

Fig. 1. Experimental tests of the optimal NC-entropic uncertainty band (25) for $(\pi^+ p)$ quantum scattering states corresponding to the nonextensivities (a) $[p = 2/3, q = 2]$, (b) $[p = 1, q = 1]$, (c) $[p = 2, q = 2/3]$. The full curves represent the optimal state PMD-SQS predictions while their errors are given by the hatched regions (see in the text).

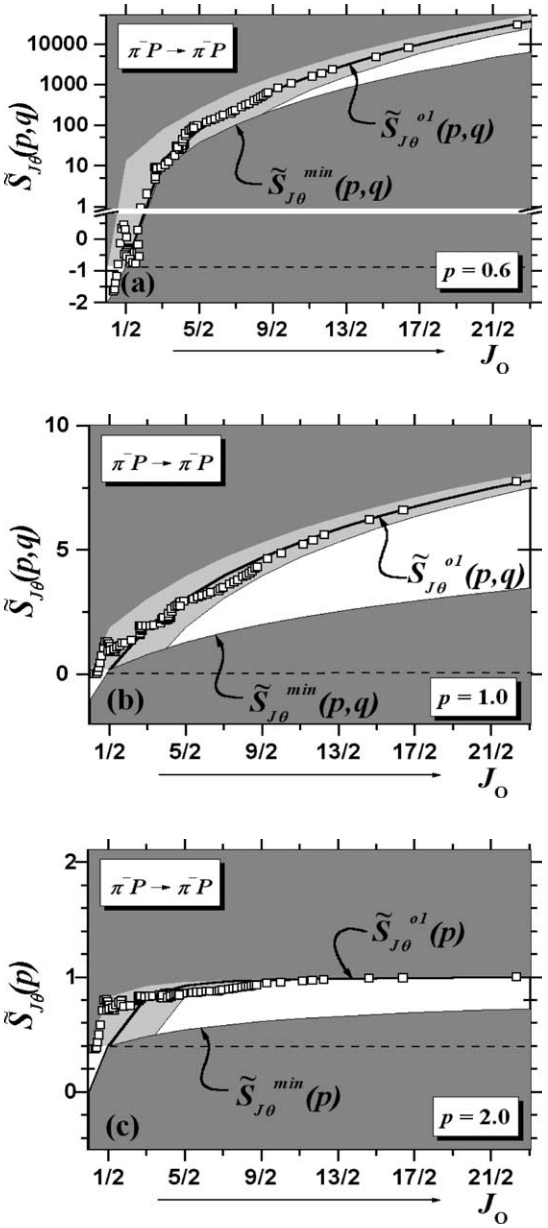

Fig. 2. Experimental tests of the optimal NC-entropic uncertainty band (25) for $(\pi^- p)$ quantum scattering states corresponding to the nonextensivities (a) $[p = 2/3, q = 2]$, (b) $[p = 1, q = 1]$, (c) $[p = 2, q = 2/3]$. The full curves represent the optimal state PMD-SQS predictions while their errors are given by the hatched regions (see in the text).

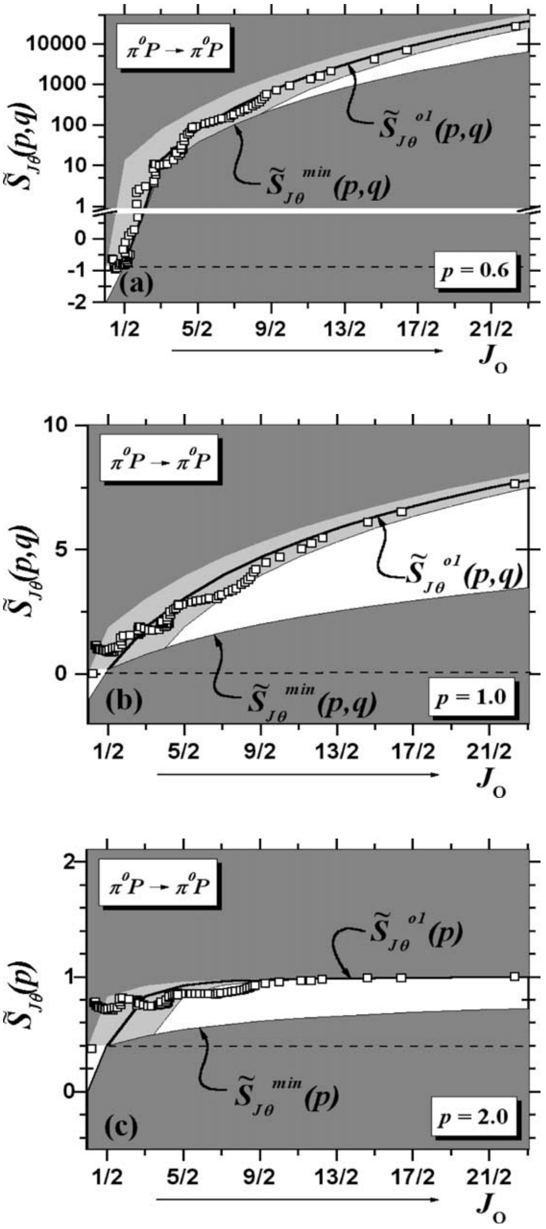

Fig. 3. Experimental tests of the optimal NC-entropic uncertainty band (25) for $(\pi^0 p)$ quantum scattering states corresponding to the nonextensivities (a) $[p = 2/3, q = 2]$, (b) $[p = 1, q = 1]$, (c) $[p = 2, q = 2/3]$. The full curves represent the optimal state PMD-SQS predictions while their errors are given by the hatched regions (see in the text).

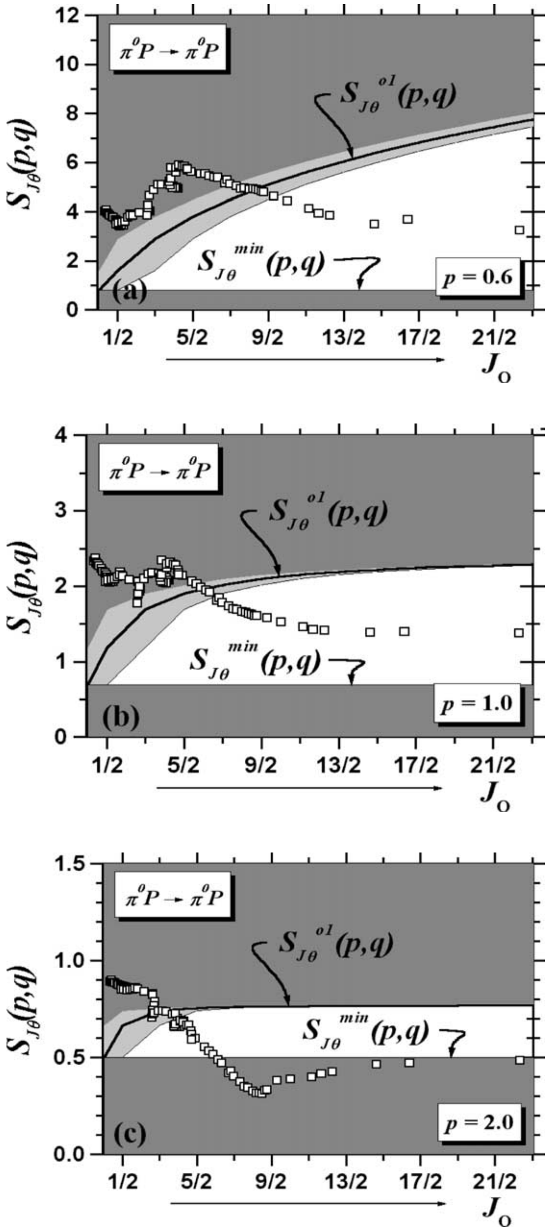

Fig. 4. Experimental tests of the entropic scattering band [9]: $(1 - 2^{1-p})/(p - 1) \leqslant S_{J\theta}(p) \leqslant S^{o1}_{J\theta}(p)$, for $(\pi^0 p)$ quantum scattering states corresponding to the nonextensivities (a) $p = 2/3$, (b) $p = 1$, (c) $p = 2$. The full curves represent the optimal state PMD-SQS predictions while their errors are given by the hatched regions (see in the text).

entropies, for example with the values $S_{J\theta}(p)$ for $\pi^0 p$ scattering presented in Fig. 4. Then, we see that only $\overline{S}_{J\theta}[p, q = p/(2p-1)]$ are in agreement with the optimal joint entropies $\overline{S}^{o1}_{J\theta}[p, q = p/(2p-1)]$ proved in our Letter while the usual $S_{J\theta}(q)$ joint entropies are far from the equilibrium values: $S^{o1}_{J\theta}(t), t = p, q$, and in some cases violate the entropic uncertainty band. These results allow us to conclude that new entropy $\overline{S}_{J\theta}[p, q = p/(2p-1)]$ is the best candidate for description of entropic uncertainty relations when the nonextensivity conjugation is taken into account.

In conclusion, by introduction of the nonextensivity conjugated joint entropy $\overline{S}_{J\theta}(p, q)$ (11), in this Letter we proved *new entropic uncertainty relations* [see the lower bound (25)] as well as *new entropic uncertainty bands* [see NC-EUB (25)]. Moreover, we obtained not only consistent experimental tests of the optimal NC-EUR and NC-EUB, but also strong experimental evidences for the *principle of minimum distance in space of quantum states* (PMD-SQS) [12,13].

Finally, we note that further investigations and extension of these results to the high-energy production phenomena (see Refs. [15–17]) are needed, not only since the the nonextensivity parameter q can be regarded as a kind of unification of all processes responsible for the actual nonextensivity into a single class, but also, since the strong nonextensive conjugated statistical behavior of the quantum states can be an *universal signature* for the saturation of (PMD-SQS)-quantum equilibrium limit in all phenomena.

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